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GLOBAL SOLUTIONS TO THE SEMILINEAR WAVE EQUATION FOR LARGE SPACE DIMENSIONS

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Consider the semilinear wave equation

$$\Box u = F(u), \tag{1}$$

where $F(u) = O(|u|^{\lambda})$ near $|u| = 0$ and $\lambda > 1$. Here and below $\Box$ denotes the d'Alembertian on $\mathbf{R}^{n+1}$.

For this semilinear wave equation W.Strauss proposed in [17] the conjecture that the existence of global solution of the corresponding Cauchy problem with small initial data depends essentially on a critical value $\lambda_0(n)$ for the non linearity, namely $\lambda_0(n)$ is the positive root of the equation

$$(n-1)\lambda^2 - (n+1)\lambda - 2 = 0. \tag{2}$$

More precisely, for the subcritical case ($1 < \lambda < \lambda_0(n)$) the conjecture asserts that the solution with small initial data blows up in finite time, while an existence result is expected for the supercritical case ($\lambda > \lambda_0(n)$) .

Here below we shall make a brief review of the results concerning this conjecture.

The case $n = 3$ was studied by F.John in the pioneer work [6]. The critical value for this case is $\lambda_0(3) = 1 + \sqrt{2}$.

For $n = 2$ a proof of the conjecture was given by R.Glassey ([4], [5]). A blow-up result for arbitrary space dimensions when $1 < \lambda < \lambda_0(n)$ was established by T.Sideris [16].

The critical values $\lambda = \lambda_0(n)$ were studied by J.Schaeffer in [15] for $n = 2, 3$. A simplified proof was found by H.Takamura [24].

Another interesting effect is the influence of the decay rate of the initial data on the existence of global solutions. In this case the solution might blow-up in finite time when the initial data decay very slowly at infinity even in the supercritical case when $\lambda > \lambda_0(n)$. For the case $n = 3$ this effect was established by F.Asakura [3] for the supercritical case. The critical cases for $n = 2, 3$ were studied by K.Kubota [13] , K.Tsutaya [25], [26], [27], R.Agemi and H.Takamura [2]. For the case $n \geq 4$ and supercritical non linearity the blow-up result for slowly decaying initial data is due to H.Takamura [23].

On the other hand, the existence part of the conjecture of W.Strauss for $n > 3$ is also very actively studied in the recent years.

Y. Zhou [28] has found a complete answer for $n = 4$ by using suitable weighted Sobolev estimates and the method developed by S.Klainerman [7], [8], [9] for proving the existence of small amplitude solutions.

The existence of a global solution for the case $\lambda = (n+3)/(n-1)$ was established by W.Strauss [19] by the aid of the conformal methods and the classical Strichartz inequality [20], [21], [22].

Another partial answer was given by R.Agemi, K.Kubota, H. Takamura in [1] for a special class of integral non linearity in (1). The approach in this work follows the approach of F.John.

A complete proof of the conjecture of W.Strauss for spherically symmetric initial data was found by H.Kubo [12] (see also [10], [11]).

By using different estimates H.Lindblad and C.Sogge [14] obtained a similar result as well as the existence of solutions in the supercritical case, non spherically symmetric initial data and space dimensions $n \leq 8$.

Our purpose in this talk shall be the announce of a result concerning the Cauchy problem

$$\begin{gathered} \Box u = F(u), \\ u(0,x) = \varepsilon f \quad , \quad \partial_t u(0,x) = \varepsilon g, \end{gathered} \tag{3}$$

where f, g are compactly supported smooth functions such that

$$\operatorname{supp} f \cup \operatorname{supp} g \subseteq \{|x| \leq R\}, \tag{4}$$

while ε is a sufficiently small positive number. For the nonlinear function $F(u)$ we shall assume that $F(u) \in C^0$ near $u = 0$ and for some $\lambda > 1$ satisfies

$$\begin{gathered} |F(u)| \leq C|u|^\lambda \quad , \\ |F(u) - F(v)| \leq C|u-v|(|u|^{\lambda-1} + |v|^{\lambda-1}) \end{gathered} \tag{5}$$

near $u, v = 0$.

Our goal shall be to examine the existence of global solution to (3) for

$$\lambda_0(n) < \lambda < \frac{n+3}{n-1}, \tag{6}$$

where $\lambda_0(n)$ is the positive root of (2). For this case we have the following

Theorem 1 *Suppose the assumptions (4) , (5) and (6) are fulfilled with $\lambda_0(n)$ being the positive root of the equation*

$$(n-1)\lambda^2 - (n+1)\lambda - 2 = 0. \tag{7}$$

Then there exists $\varepsilon_0 > 0$ so that for $0 < \varepsilon < \varepsilon_0$ the Cauchy problem (3) admits a global solution.

The solution belongs to a Banach space of type

$$u \in L^q_{\alpha,\beta}(\mathbf{R}^{n+1}_+),$$

where $L^q_{\alpha,\beta}(\mathbf{R}^{n+1}_+)$ denotes the Banach space of all measurable functions with finite norm

$$\|\tau_+^{\alpha}\tau_-^{\beta}u\|_{L^q(\mathbf{R}^{n+1}_+)}.$$

Here and below $\tau_\pm = 1 + |t \pm |x||$ are the weights associated with the characteristic surfaces of the wave equation.

The result of the above theorem shows that the conjecture of W.Strauss is valid for arbitrary space dimensions $n \geq 2$ even in the case of non spherically symmetric initial data.

The main idea to establish the above result is the application of a weighted estimate for the inhomogeneous wave equation

$$\Box u = F, \tag{8}$$

with zero initial data. For simplicity we shall assume that the supports of u and F lie in the light cone, that is

$$\mathrm{supp} F(t,x) \subset \{|x| \leq t + R\}. \tag{9}$$

The key to prove Theorem 1 is the following weighted estimate.

Theorem 2 *Suppose $1 < p, q < \infty$ satisfy*

$$\begin{gathered} \frac{1}{q} < \frac{1}{p} \ , \ \frac{1}{q} + \frac{1}{p} \leq 1, \\ \frac{n-3}{2} < \frac{n}{q} - \frac{1}{p}, \end{gathered} \tag{10}$$

while the parameters $\alpha, \beta, \gamma, \delta$ satisfy

$$\begin{aligned} \alpha &< \frac{n-1}{2} - \frac{n}{q}, \\ \frac{n-1}{2p} - \frac{n+1}{2q} < \beta = \gamma - \frac{n+1}{2} + \frac{n}{p} - \frac{1}{q} &< \frac{n-1}{2} - \frac{n}{q}, \\ \delta &> 1 - \frac{1}{p}. \end{aligned} \tag{11}$$

Then the solution u satisfies the estimate

$$\|\tau_+^{\alpha}\tau_-^{\beta}u\|_{L^q(\mathbf{R}_+^{n+1})} \leq C\|\tau_+^{\gamma}\tau_-^{\delta}F\|_{L^p(\mathbf{R}_+^{n+1})}, \tag{12}$$

where $\tau_{\pm} = 1 + |t \pm |x||$ *and* $\mathbf{R}_+^{n+1} = \{(t,x) \in \mathbf{R}^{n+1} : t \geq 0\}$.

This estimate can be considered as a generalization of the Strichartz estimate and the estimates used by F.John in [6].

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